JOURNAL OF BASIC SCIENCE AND ENGINEERING

STABILITY ANALYSIS ON ABSTRACT SECOND-ORDER FUNCTIONAL IMPULSIVE DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY

C.Gowrisankar and Dr.K.Malar
Department of Mathematics, Erode Arts and Science College, Rangampalayam, Erode, Tamilnadu, India.

Abstract

The paper investigates Hyers-Ulam stability for impulsive second-order abstract functional differential systems with state-dependent delay. It derives its results from the framework of significantly constant cosine families of linear operators. The work establishes a number of definitions of Hyers-Ulam stability by utilising Gronwall's type inequality and Lipschitz criteria. Theoretical derivations are proved by an example of a wave equation that successfully explains how the acquired results can be applied in practise. The study contributes to the understanding of stability occurrences in a specific class of functional differential systems by addressing complicated systems with impulsive dynamics and state-dependent delay, as well as by utilising rigorous mathematical techniques.

Keywords: State-dependent delay, Functional differential equation, Impulsive systems, Hyers-Ulam-Rassias Stability.

1. Introduction

The study of impulsive second-order abstract functional differential systems with state-dependent delay aims to develop conditions for stable solutions in the surface of impulsive effects and delay. Stability analysis includes determining solution existence, uniqueness, continuity, and boundedness in the presence of impulses. The investigation of Hyers-Ulam stability improves understanding of solution behaviour and robustness in setting involving impulsive effects and state-dependent delay. The results of such investigations provide essential insights applicable to the design and control of dynamical systems in a variety of disciplines. As a result, investigating Hyers-Ulam stability within these systems is a significant challenging research field with far-reaching applications. This subject of study links theoretical exploration with real-world application, emphasising its importance in understanding and manipulating complex dynamical behaviours, especially under difficult settings.

An extremely important investigation concerns the Hyers-Ulam stability, which was first investigated using functional equations. This narrative began with the Ulam issue, which Ulam proposed at Wisconsin University in 1940. Hyers, who pioneered this discussion, offered the first solution in 1941, establishing the concept of stability for functional equations.

Since then, an array of mathematicians[1,2] have worked tirelessly to investigate the stability of various functional equations and their pragmatic applications, as documented in Rassia's work in 2014.Wang et.al.[18] expanded the scope of Hyers-Ulam stability to include ordinary linear differential equation in 1993, describing it as the presence of an approximative solution close to the precise solution. Finally, numerous worldwide authors[5,6,9,11,12] have thoroughly examined the Hyers-Ulam stability, including its first and higher order differential equation variations, contributing to a full understanding of this paradigm.

Delay differential equations with state-dependent delays develop naturally in a variety of modelling contexts. For example, they are used to simulate infection and illness transmission, immunological responses, and respiration. These delays are caused by the time it takes to accumulate sufficient infection or antigen concentration.

In mathematical modelling, the differential equations with impulses theory have received a lot of popularity. Many real-world processes undergo sudden state shifts at critical points. Within the system, these processes experience transitory disturbances known as impulsive effects. Numerous researchers, including Samoilenko and Perestyuk[13], Lakshmikantham, et.al.[7], Rogovchenko[10] and Wang et.al.[16], have descending into the world of differential equations with impulses in recent years, indicating an increasing interest in this area.

Akbar Zada, Luqman Alam, Jiafa Xu, and Wei Dong investigated a damped second-order abstract impulsive differential equation system. Their goal was to explore its Hyers-Ulam stability. They achieved this by employing a specific version of Gronwall's inequality and Lipschitz conditions.

In 2018, E. Hernandez et.al.[3] established the existence and distinctiveness of the following solution:

$$\vartheta''(\tau) = A\vartheta(\tau) + \mathfrak{F}\left(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)}\right), \;\; \tau \,\varepsilon\, [0,b] \tag{1.1}$$

$$\vartheta(0) = \varrho \in C([-\gamma,0];X), \vartheta'(O^+) = x \in X \tag{1.2}$$

$$\Delta\vartheta(\tau_k) = I_k\left(\vartheta_{\tau_k}\right), \;\; k = 1,2,3,\dots n \tag{1.3}$$

$$\Delta\vartheta'(\tau_k) = J_k\left(\vartheta_{\tau_k}\right), \;\; k = 1,2,3,\dots n \tag{1.4}$$

The objective of the paper is to identify the criteria that guarantee Hyers-Ulam stability for the impulsive second-order abstract functional differential systems with state-dependent delay. The main goal is to apply rigorous mathematical approaches such as fixed-point theory, Lyapunov functional methods, and comparison principles. These methods are used to determine appropriate stability conditions. The investigation includes both continuous and discrete impulsive scenarios, with stability analysis performed in each case. The results of the gained stability improve understanding of the dynamic behaviour of impulsive second-order abstract functional differential systems with state-dependent delay. These outcomes have significance for the structure and control of real-world systems involving impulsive effects and state-dependent delays. The findings have

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

implications for designing and maintaining real-world systems in which impulsive affects and delays based on the system's state are relevant. The beginning part of this research study gives the necessary symbol representations, definitions, and outcomes for the present research.

2. Basic preliminaries

In this part, several definitions and statements may be utilized in the basic results. Throughout the research paper, we employed $(X, \|.\|)$ is a Banach space and $A: D(A) \subset X \to X$ is the infinitesimal generator of $(\mathcal{C}(\tau)_{\tau\in\mathbb{R}})$ of linear bounded operators on X. It is denoted that $(p(\tau))\_(T \in R)$ the sine functions with $\left(\mathcal{C}(\tau)\right)_{\tau\in\mathbb{R}}$, is described by $\wp(\tau)x = \int_0^\tau \mathcal{C}(\tau)x d\tau$, for $x \in X$ and $\tau \in R$. Furthermore, M and N are significant constant to the extent that $\| \mathcal{C}(\tau) \| \leq M$ and $\| \wp(\tau) \| \leq N$, for each $\tau \in I$. It is assumed that $\varphi \in B$ and that $\sigma : I \times B \to (-\infty, b]$ is continuous function. The symbol E indicates the space of vectors $x \in X$ for which $\mathcal{C}(.)x$ is of C^1.

Definition 2.1 A function $\vartheta(.)$ is said to be a mild solution of (1.1)–(1.4) if $\vartheta_0 = \varrho, \vartheta_{\sigma(s,x_s)} \in B$ for every $s \in I$ and

$$\vartheta(\tau) = \mathcal{C}(\tau)\varrho + \wp(\tau) + \int_0^\tau \wp\left(\tau - s\mathfrak{F}\left(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)}\right)\right) ds + \sum_{\tau_k < \tau} \mathcal{C}(\tau - \tau_k) I_k(\vartheta(\tau_k) + \sum_{\tau_{k<\tau}} \wp\,(\tau - \tau_k) J_k(\vartheta(\tau_k) \qquad (2.5)$$

In the section, the following assumptions are examined:

$\mathbf{H}_{\boldsymbol{\varphi}}$ Let $\mathcal{R}(\sigma^-) = \{\sigma(s, \psi) : (s, \psi) \in I \times B, \sigma(s, \psi) \leq 0\}$. The function $t \to \varphi_t$ is well defined from $\mathcal{R}(\sigma^-)$ into B, there is a continuous and bounded function $J^\varphi : \mathcal{R}(\sigma^-) \to \mathbb{R}$ such that as $\| \varphi_t \|_B \leq J^\varphi(t) \| \varphi \|_B$ for every $t \in \mathbb{R}(\sigma^-)$.

$\mathbf{H}_1$ The function f: $I \times B \to X$ full fills the following conditions:

(i) Let $x : (-\infty, b] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. the function t $\to f(t, x_{\sigma(t,x_t)})$ is measurable on I and the function $t \to f(s, x_t)$ is continuous on $\mathcal{R}(\sigma^-) \cup I$ for every $s \in I$.

(ii) For each $t \in I$, the function $f(t,\cdot) : B \to X$ *is continuous*.

(iii) There is an integrable function $h: I \to [0, \infty)$ and a continuous non-decreasing function $W: [0, \infty) \to (0, \infty)$ such that

$$\| \mathfrak{F}(t, \psi) \| \leq h(t) W(\| \psi \|_B), (t, \psi) \varepsilon\, I \times B$$

(iv) There exists $L_{\mathfrak{F}} > 0$ such that

$$\| \mathfrak{F}(t, \psi_1) - \mathfrak{F}(t, \psi_2) \| \leq L_{\mathfrak{F}} \| \psi_1 - \psi_2 \|, \quad t \in I,\ \psi_1, \psi_2 \in B \qquad (2.6)$$

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

Definition 2.2 Let V be a vector space over some field K.

A function $\|\cdot\|_\beta : V \to [0,\infty)$ is called $\beta-$ norm if:

(a):$\| \varpi \|_\beta = 0$ if and only if $\varpi = 0$,

(b):$\| c\varpi \|_\beta = |\varpi|^\beta \| \varpi \|_\beta$ for each c ∈ K and u ∈ V,

(c):$\| \varpi + v \|_\beta \leq \| \varpi \|_\beta + \| v \|_\beta$.

Then, (V,$\|\cdot\|_\beta$) is known as $\beta-$normed space. A certain required condition may be utilised in research findings.

$\mathbf{H_2}$ A is the infinitesimal generator of $\left(\mathcal{C}(\tau)\right)_\tau \in R$.

$\mathbf{H_3}$ The functions $I_k, J_k : X \to X$ are continuous and there prevail positive constants L_1 and L_J such that:

$\| I_k(\varpi) - I_k(v) \| \leq L_I \| \varpi - v \|, \| J_k(\varpi) - J_k(v) \| \leq L_J \| \varpi - v \|$ (2.7) $\mathbf{H_4}$ There exists a increasing function $\varrho \in PC(I,S)$ with $\varrho(\tau) \geq 0$ and a constant c_ϱ such that, $\int_0^\tau \varrho(\tau)d\tau \leq c_\varrho \varrho(\tau)$ (2.8)

For each $\tau \in I$ and $\tau \in S$.

Remark 2.1 In the paper $y : +(-\infty, b] \to X$ is the function specified by $y(t) = \varphi(t)$ on $(-\infty, 0]$ and $y(t) = \mathcal{C}(t)\varphi(0) + \wp(t)\zeta$ for $t \in I$. Besides, $\| y \|_b, M_b, K_b, and\ J_0^\varphi$ are the constant characterised by $\| y \|_b = \sup_{s\in[0,b]} \| y(s) \|$, $\mathrm{M_b} = \sup_{s\in[0,b]} M(s)$, $K_b = \sup_{s\in[0,b]} K(s)$ and $J_0^\varphi = \sup_{t\in\mathcal{R}(\sigma^-)} J^\varphi(t)$ are discussed further.

Lemma 2.1 [24, Lemma 2.1] Let x $(-\infty, b) \to X$ be a function such that $x_0 = \varphi$ and $x|_I \in PC$. Then,

$$\| x_s \|_B \leq \left(M_b + J_0^\varphi\right) \| \varphi \|_B + K_b \sup\{\| x(\tau) \| ; \tau \in [0, \max\{0,s\}]\}, \quad s \in \mathcal{R}(\sigma^-) \cup I.$$

Lemma 2.2 It is presumed that all the inferences listed in Lemma 3.1 are accomplished. At that time the operator

$$\Gamma_\gamma(\tau) = \int_0^\tau \wp(\tau - s)\left[f\left(s, \gamma(s)\right) + \left(Du_\gamma\right)(s)\right]ds, \quad \tau \in [0,\rho] \tag{2.9}$$

Is completely continuous.

Lemma 2.3 Predict that ζ is a condensing operator on X. $If\ \zeta(A) \subset A$ is closed, bounded and convex set of X, then ζ has a fixed point in A.

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

Lemma 2.4 (Gronwall's Lemma[25]) for any $\tau \geq 0$ with

$$\vartheta(\tau) \leq q(\tau) + \int_0^\tau \mathcal{P}(s)\vartheta(s)ds + \sum_{o<\tau_k<\tau} \gamma_k \vartheta(\tau_k^-) \tag{2.10}$$

Where s, p, q $\in PC(I, \mathbb{R}^+)$, q is increasing and $\gamma > 0$, it is obtained as

$$\vartheta(\tau) \leq q(\tau)(1+\gamma_k)^k e^{\int_0^\tau p(s)ds}, \forall \gamma \in \mathbb{R}^+ \tag{2.11}$$

Where k$\in M$.

3. Ulam's Type Stability

Hernandez et.al.[(4)] establish the solution for the system:

$$\left.\begin{aligned} &\vartheta'' = A\vartheta + \mathfrak{F}(\tau, \vartheta_\sigma) \\ &\Delta\vartheta(\tau_k) = I_k(\vartheta(\tau_k)), \quad k = 1,2,3, \ldots, n \\ &\Delta\vartheta'(\tau_k) = J_k(\vartheta(\tau_k)), \quad k = 1, 2, 3, \ldots, n \\ &\vartheta(0) = \varrho, \vartheta(0) = x \end{aligned}\right\} \tag{3.12}$$

In the form

$$\vartheta(\tau) = \mathcal{C}(\tau)\varrho + \wp(\tau)x + \int_0^\tau \wp(\tau - s)\mathfrak{F}(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)})ds + \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)I_k(\vartheta(\tau_k)) + \sum_{\tau_k>\tau} \wp(\tau - \tau_k)J_k(\vartheta(\tau_k)) \tag{3.13}$$

Let $\epsilon > 0, \psi \geq 0$ & $\varrho \in PC(I, R^+)$ be the increasing functions. The below mentioned inequalities are considered

$$\begin{cases} \| r'' - \mathrm{A}r(\tau_k) - \mathfrak{F}(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)}) \| \leq \epsilon, & \tau \in I \\ \| \Delta r(\tau_k) - I_k(r(\tau_k)) \| \leq \epsilon, & \tau \neq \tau_k \\ \| \Delta r_t(\tau_k) - J_k(r(\tau_k)) \| \leq \epsilon, & \tau \neq \tau_k \end{cases} \tag{3.14}$$

And

$$\begin{cases} \| r'' - Ar(\tau_k) - \backslash\mathfrak{F}(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)}) \| \leq \epsilon, & \tau \in I \\ \| \Delta r(\tau_k) - I_k(r(\tau_k)) \| \leq \epsilon, & \tau \neq \tau_k \\ \| \Delta r_t(\tau_k) - J_k(r(\tau_k)) \| \leq \epsilon & \tau \neq \tau_k \end{cases} \tag{3.15}$$

Remark 3.1 It has a direct impact of inequality (3.14) that a function $\gamma \in Z$ is a solution for the inequality (3.14), if and only if there are G$\in C^2$(I , X), $g_1 \in C(I, X)$ and $g_2 \in C_1(I, X)$ like

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

$$\begin{cases} \| G(\tau) \| \leq \epsilon, \| g_1(\tau) \| \leq \epsilon \ \ \& \ \| g_2(\tau) \leq \epsilon, \tau \in I \\ r''(\tau) = Ar(\tau) + \mathfrak{F}\big(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)}\big) + G(\tau), \tau \in I, \tau \neq \tau_k, k = 1, 2, \dots n \\ r(0) = \varrho + G(\tau), r'(0) = x + G(\tau) \\ \Delta r(\tau_k) = I_k\big(r(\tau_k)\big) + g_1(\tau_k), k = 1,\ 2, \dots . n \\ \Delta r'(\tau_k) = J_k(r(\tau_k)) + g2(\tau_k), k = 1, 2, \dots n \end{cases} \tag{3.16}$$

Definition 3.1. The system (3.12) is Hyers-Ulam (HU) stable if abides $\vartheta(K_1, L_1, L_\zeta) > 0$ likewise for each $\epsilon > 0$ and every solution $\Upsilon \in Z$ of the inequality (4.3), a solution $\vartheta \in Z$ of equation (3.12) is found, such as

$$\| \Upsilon(\tau) - \vartheta(\tau) \| \leq \vartheta\big(k_1, L_1, L_\zeta\big)\epsilon, \quad \tau \in I \tag{3.17}$$

Definition 3.2 The equation (3.12) is Hyers-Ulam-Rassias (HUR) stable in connection with (ϱ, ψ) if it exists $\vartheta(K_1, L_1, L_\zeta, \varrho) > 0$ such as for each $\epsilon > 0$ and every solution $\Upsilon \in Z$ of the inequality (4.4), a solution $\vartheta \in Z$ of equation (3.12) is found, and said that

$$\| \Upsilon(\tau) - \vartheta(\tau) \| \leq \vartheta\big(K_1, L_1, L_\zeta\big)\epsilon(\varrho(\tau) + \tau\psi), \quad \tau \in I \tag{3.18}$$

Definition 3.3 The equation (3.12) $\beta -$ HUR stable with respect to $(\varrho^\beta, \psi^\beta)$ if it exists $\vartheta(K_1, L_1, L_\zeta, \varrho, \psi) > 0$ such as for each $\epsilon > 0$, and every solution $\Upsilon \in Z$ of the inequality (4.4), there is a solution $\vartheta \in Z$ of equation (3.12), like

$$\| \Upsilon(\tau) - \vartheta(\tau) \| \leq \vartheta\big(K_1, L_1, L_\zeta, \varrho, \psi\big)\epsilon(\varrho(\tau) + \tau\psi), \tau \in I \tag{3.19}$$

Theorem 3.1 If inferences $[H_\varphi]$ and $[H_1]$ – $[H_3]$ are satisfied, then the equation (3.12) is Hyers Ulam stable concerning ϵ.

Proof: Based on the remark 3 .1, it is said that solution of the system (3.16) is

$$\gamma(\tau) = \mathcal{C}(\tau)\varrho + \wp(\tau)x + \int_0^\tau \wp(\tau - s)\mathfrak{F}\big(\tau, \gamma_{\sigma(\tau,\gamma_\tau)}\big)ds + \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)I_k(\gamma(\tau_k)) + \sum_{\tau_k<\tau} \wp(\tau - \tau_k)J_k(\gamma(\tau_k))$$

$$= \mathcal{C}(\tau)r(0) + \wp(\tau)r'(0\,) + \int_0^\tau \wp(\tau - s)\mathfrak{F}\big(\tau, \gamma_{\sigma(\tau,\gamma_\tau)}\big)ds$$

$$+ \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)I_k\big(\gamma(\tau_k)\big) + \sum_{\tau_k<\tau} \wp(\tau - \tau_k)J_k\big(\gamma(\tau_k)\big)$$

$$= \mathcal{C}(\tau)[\varrho + G(\tau)] + \wp(\tau)[x + G(\tau)] + \int_0^\tau \wp(\tau - s)[\mathfrak{F}\big(\tau, \gamma_{\sigma(\tau,\gamma_\tau)}\big) + G(s)]ds$$

$$+ \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)\big[I_k\big(\gamma(\tau_k)\big) + g_1(\tau_k)\big] + \sum_{\tau_k<\tau} \wp(\tau - \tau_k)[J_k\big(\gamma(\tau_k)\big)g_2(\tau_k)]$$

Let γ be the solution of inequality (3.14). Further for every $\tau \in I$, it is obtained

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

$$\| \gamma(\tau) - \mathcal{C}(\tau)\varrho - \wp(\tau)x - \int_0^\tau \wp(\tau - s)\mathfrak{F}(\tau, \gamma_{\sigma(\tau,\gamma_\tau)})ds - \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)I_k(\gamma(\tau_k)) - \sum_{\tau_k<\tau} \wp(\tau - \tau_k)J_k(\gamma(\tau_k)) \| \leq M\epsilon + N\epsilon + N\epsilon \int_0^\tau ds + \epsilon M\tau + \epsilon N\tau$$

$$\leq \epsilon[M + N] + \epsilon N\tau + \epsilon M\tau + \epsilon N\tau$$

$$\leq \epsilon(M_0 + M_\tau + 2N_\tau) \tag{3.20}$$

Henceforth, for each $\tau \in I$ is obtained as

$$\| \gamma(\tau) - \vartheta(\tau) \| \leq \| \mathcal{C}(\tau)G(\tau) \| + \| \wp(\tau)G(\tau) \| + \| \int_0^\tau \wp(\tau - s)G(s)ds \| + \| \int_0^\tau \wp(\tau - s)[\mathfrak{F}(\tau, \gamma_{\sigma(\tau,\gamma_\tau)}) - \mathfrak{F}(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)})]ds \| + \| \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)[I_k(\gamma(\tau_k)) - I_k(\vartheta(\tau_k))] \| + \| \sum_{\tau_k<\tau} \wp(\tau - \tau_k)[J_k(\gamma(\tau_k)) - J_k(\vartheta(\tau_k))] \| + \| \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)g_1(\tau_k) \| + \| \sum_{\tau_k<\tau} \wp(\tau - \tau_k)g_2(\tau_k) \| \leq (M + N)\epsilon + \epsilon N\tau + N \int_0^\tau L_{\mathfrak{F}}(s) \| \gamma(s) - \vartheta(s) \| ds + M \sum_{\tau_k<\tau} L_1 \| \gamma(\tau_k) - \vartheta(\tau_k) \| + N \sum_{\tau_k<\tau} L_J \| \gamma(\tau_k) \| + \epsilon M\tau + \epsilon N\tau$$

$$\leq \epsilon(M_0 + M_\tau + 2N\tau) + N \int_0^\tau L_{\mathfrak{F}}(s) \| \gamma(s) - \vartheta(s) \| ds$$

$$2M_1 \sum_{\tau_k<\tau} L_1 \| \gamma(\tau_k) - \vartheta(\tau_k) \| \tag{3.21}$$

Where $M_1 = \max\{M, N\}\, and\ L_1 = \max\{L_I, L_J\}$.

Using lemma 2.4,

$$\| \gamma(\tau) - \vartheta(\tau) \| \leq \epsilon[M_0 + M_\tau + 2N_\tau][1 + 2M_1L_1]^m e^{N[\int_0^\tau L_{\mathfrak{F}}(s)ds]}$$

$$\leq \epsilon V(K_1, L_1, L_{\mathfrak{F}}) \tag{3.22}$$

Where $\mathrm{V}(K_1, L_1, L_F) = [M_0 + M_\tau + 2N_\tau][1 + 2M_1L_1]^m e^{N[\int_0^\tau L_{\mathfrak{F}}(s)ds]}$ Therefore, the equation (3.12) is Hyers Ulam stable.

Remark 3.2

$$\begin{cases} \| G(\tau) \| \leq \epsilon\varrho(\tau), \quad \| g_1(\tau) \| \leq \epsilon\psi \quad \& \ \| g_2(\tau) \| \leq \epsilon\psi \\ r''(\tau) = Ar(\tau) + \mathfrak{F}(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)}) + G(\tau), \quad \tau \in I, \quad \tau \neq \tau_k \\ r(0) = \varrho + G(\tau), \quad r'(0) = x + G(\tau) \\ \Delta r(\tau_k) = I_k(r(\tau_k)) + g_1(\tau_k) \\ \Delta r'(\tau_k) = J_k(r(\tau_k)) + g_2(\tau_k) \end{cases} \tag{3.23}$$

Theorem 3.2 If assumptions $[H_1]$ - $[H_3]$ are redeemed, in addition the equation (3.12) is HUR stable concerning (ϱ, ψ).

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

Proof: If γ is a solution of the inequality (3.15) and ϑ is the unique solution of the system (3.12), which is mentioned in (3.13). Based on the Remark (3.2), the solution of the system (3.23) is referred as

$$\gamma(\tau) = \mathcal{C}(\tau)\varrho + C(\tau)G(\tau) + \wp(\tau)x + \wp(\tau)G(\tau) + \int_0^\tau \wp(\tau - s)F(\tau, \gamma_\sigma)ds$$

$$+ \int_0^\tau \wp(\tau - s)G(s)ds + \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)I_k(\gamma(\tau_k)) + \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)g_1(\tau_k)$$

$$+ \sum_{\tau_k<\tau} \wp(\tau - \tau_k)I_k(\gamma(\tau_k)) + \sum_{\tau_k<\tau} \wp(\tau - \tau_k)g_2(\tau_k) \tag{3.24}$$

If γ is a solution of (3.15), then for each $\tau \in I$ is obtained as

$$\| \gamma(\tau) - \mathcal{C}(\tau)\varrho - \wp(\tau)x - \int_0^\tau S(\tau - s)\mathfrak{F}(\tau, \gamma_\sigma)ds - \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)I_k(\gamma(\tau_k)) -$$

$$\sum_{\tau_k-\tau} \wp(\tau - \tau_k)J_k(\gamma(\tau_k)) \| \leq \| \mathcal{C}(\tau)G(\tau) \| + \| \wp(\tau)G(\tau) \|$$

$$+ \| \int_0^\tau \wp(\tau - s)G(s)ds \| + \| \sum_{\tau_k-\tau} \mathcal{C}(\tau - \tau_k)g_1(\tau_k) \| \ \| \sum_{\tau_k-\tau} \wp(\tau - \tau_k)g_2(\tau_k) \|$$

$$\leq M\epsilon\varrho(\tau) + N\epsilon\varrho(\tau) + N \| \int_0^\tau G(s)ds \| + M\tau\epsilon\psi + N\tau\epsilon\psi$$

$$\leq (M + N)\epsilon\varrho(\tau) + N \| \int_0^\tau \epsilon\varrho(s)ds \| + (M + N)\tau\epsilon\psi$$

$$\leq M_0\epsilon\varrho(\tau) + N\epsilon C_\varrho\varrho(\tau) + M\tau\epsilon\psi + N\tau\epsilon\psi$$

$$\leq \epsilon[\varrho(\tau) + \tau\psi][M_0 + NC_\varrho].$$

Hence, for each $\tau \in I$, is obtained as

$$\| \gamma(\tau) - \vartheta(\tau) \| \leq \| \mathcal{C}(\tau)G(\tau) \| + \| \wp(\tau)G(\tau) \| + \| \int_0^\tau \wp(\tau - s)G(s)ds \|$$

$$+ \int_0^\tau \wp(\tau - s)[\mathfrak{F}(\tau, \gamma_{\sigma(\tau,\gamma_\tau)}) - \mathfrak{F}(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)})]ds \|$$

$$+ \| \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)[I_k(\vartheta(\tau_k))] \| + \| \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)g_1(\tau_k) \|$$

$$+ \| \sum_{\tau_k<\tau} \wp(\tau - \tau_k)[J_k(\gamma(\tau_k)) - J_k(\vartheta(\tau_k))] \| + \| \sum_{\tau_k<\tau} \wp(\tau -)g_2(\tau_k) \|$$

$$\leq [M + N]\epsilon\varrho(\tau) + N \| \int_0^\tau \epsilon\varrho(s)ds \| + M\tau\epsilon\psi + N\tau\epsilon\psi +$$

$$N \int_0^\tau L_{\mathfrak{F}}(s) \| \gamma(s) - \vartheta(s) \| ds + 2M_1 \sum_{\tau_k<\tau} L_1 \| \gamma(\tau_k) - \vartheta(\tau) \| \leq$$

$$\epsilon[\varrho(\tau) + \tau\psi][M_0 + Nc_\varrho] + 2M_1 \sum_{\tau_k<\tau} L_1 \| \gamma(\tau_k) - \vartheta(\tau_k)$$

$$+N \int_0^\tau L_{\mathfrak{F}}(s) \| \gamma(s) - \vartheta(s) \| ds, \tag{3.25}$$

Where $M_1 = \max(M, N)$ *and* $L_1 = \max(L_1, L_J)$. Through using lemma 2.4, is fulfilled as

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

$$\| \gamma(\tau) - \vartheta(\tau) \| \leq \epsilon[\varrho(\tau) + \tau\psi]\left[M_0 + Nc_{\varrho}\right][1 + 2M_1L_1]^m e^{N\int_0^\tau L_{\mathfrak{F}}(s)ds}$$

$$\leq \left[k_1, L_1, L_{\mathfrak{F}}, \varrho\right]\epsilon[\varrho(\tau) + \tau\psi], \tag{3.26}$$

Where $V[k_1, L_1, L_F, \varrho] = \left[M_0 + Nc_{\varrho}\right][1 + 2M_1L_1]^m e^{N\int_0^\tau L_{\mathfrak{F}}(s)ds}$.

Therefore, the system is HUR stable in regarded to $[\varrho, \psi]$.

Theorem 3.3 If assumptions [H1]-[H3] and definition (2.2) are achieved, then equation (3.12) is $\beta - HUR$ stable concerning $(\varrho^\beta, \psi^\beta)$.

Proof: Let γ be a findings for the inequality (3.15) and ϑ be a unique solution of the system (3.12), which is mentioned in (3.13). Based on the Remark 3.2, the solution of the system (3.15) is

$$\gamma(\tau) = \mathcal{C}(\tau)[\varrho + G(\tau)] + \wp(\tau)[x + G(\tau)] + \int_0^\tau \wp(\tau - s)[\mathfrak{F}(\tau, \gamma_{\sigma(\tau,\gamma_\tau)}) + G(s)]ds$$

$$+\sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)[I_k(\gamma(\tau_k)) + g_1(\tau_k)] + \sum_{\tau_k<\tau} \wp(\tau - \tau_k)[J_k(\gamma(\tau_k)) + g_2(\tau_k)]$$

If γ is a solution for (3.15), then for each $\tau \in I$, is fulfilled as

$$\| \gamma(\tau) - \mathcal{C}(\tau)\varrho - \wp(\tau)x - \int_0^\tau \wp(\tau - s)\mathfrak{F}(\tau, \gamma_{\sigma(\tau,\gamma_\tau)})ds - \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)I_k(\gamma(\tau_k)) -$$

$$\sum_{\tau_k<\tau} \wp(\tau - \tau_k)J_k(\gamma(\tau_k)) \| \leq \| \mathcal{C}(\tau)G(\tau) + \| \wp(\tau).G(\tau) \| + \| \int_0^\tau \wp(\tau - s)G(s)ds \|$$

$$\| \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)g_1(\tau_k) \|$$

$$\| \sum_{\tau_k<\tau} \wp(\tau - \tau_k)g_2(\tau_k) \|$$

$$\leq M\epsilon\varrho(\tau) + N\epsilon\varrho(\tau) + N \| \int_0^\tau G(s)ds \| + M\tau\epsilon\psi + N\tau\epsilon\psi$$

$$\leq (M + N)\epsilon\varrho(\tau) + N \| \int_0^\tau \epsilon\varrho(s)ds \| + (M + N)\tau\epsilon\psi$$

$$\leq \epsilon[\varrho(\tau) + \tau\psi][M_0 + Nc_{\varrho}].$$

Hence, for each $\tau \in I$, is obtained as

$$\| \gamma(\tau) - \vartheta(\tau) \|^\beta \leq [\| \mathcal{C}(\tau)G(\tau) \| + \| \wp(\tau)G(\tau) \| + \| \int_0^\tau \wp(\tau - s)G(s)ds \|$$

$$+\| \int_0^\tau \wp(\tau - s)[\mathfrak{F}(\tau, \gamma_{\sigma(\tau,\gamma_\tau)}) - \mathfrak{F}(\tau, \vartheta_{\sigma(\tau,\vartheta_\tau)})]\, ds$$

$$+\| \sum_{\tau_k<\tau} \mathcal{C}(\tau - \tau_k)\left[I_k(\gamma(\tau_k)) - I_k(\vartheta(\tau_k))\right] \| + \sum_{\tau_k<\tau} \mathcal{C}(\tau, \tau_k)g_1(\tau_k) \|$$

$$+ \sum_{\tau_k<\tau} \wp(\tau - \tau_k)\left[J_k(\gamma(\tau_k)) - J_k(\vartheta(\tau_k))\right] \| + \| \sum_{\tau_k<\tau} \wp(\tau - \tau_k)g_2(\tau) \|]^\wedge\beta$$

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

$$\leq [M\epsilon\varrho(\tau) + N\epsilon\varrho(\tau) + N \parallel \int_0^\tau G(s)ds \parallel + M\tau\epsilon\psi + N\tau\epsilon\psi \ + N\int_0^\tau L_{\mathfrak{F}}(\tau) \parallel \gamma_\sigma - \vartheta_\sigma \parallel ds + M\sum_{\tau_k<\tau} L_1 \parallel \gamma(\tau_k) - \vartheta(\tau_k) \parallel$$

$$N\sum_{\tau_k>\tau} L_j \parallel \gamma(\tau_k) - \vartheta(\tau_k) \parallel]\hat{}\beta$$

$$\leq [[M+N]\epsilon\varrho(\tau) + N \parallel \int_0^\tau \epsilon\varrho(s)\, ds \parallel + (M+N)\tau\epsilon\psi + N\int_0^\tau L_{\mathfrak{F}}(s) \parallel \gamma(s) - \vartheta(s) \parallel ds + 2M_1\sum_{\tau_k<\tau} L_1 \parallel \gamma(\tau_k) - \vartheta(\tau_k) \parallel]\hat{}\beta$$

$$\leq [\epsilon[\varrho(\tau) + \tau\psi]\left[M_0 + Nc_\varrho\right] + 2M_1\sum_{\tau_k<\tau} L_1 \parallel \gamma(\tau_k) - \vartheta(\tau_k) \parallel \ + N\int_0^\tau L_{\mathfrak{F}}(s) \parallel \gamma(s) - \vartheta(s) \parallel ds]\hat{}\beta$$

$$\leq \left[\epsilon[\varrho(\tau) + \tau\psi]\left[M_0 Nc_\varrho\right]\right]^\beta + \left[2M_1\sum_{\tau_k>\tau} L_1 \parallel \gamma(\tau_k) - \vartheta(\tau_k)\right]^\beta$$

$$\left[N\int_0^\tau L_{\mathfrak{F}}(s) \parallel \gamma(s) - \vartheta(s) \parallel ds\right]^\beta,$$

Where, $M_1 = \max\{M, N\}$ and $L_1 = \max\{L_I, L_J\}$. Therefore,

$$\parallel \gamma(\tau) - \vartheta(\tau) \parallel^\beta \ \leq \ \left[\epsilon[\varrho(\tau) + \tau\psi]\left[M_0 + Nc_\varrho\right]\right]^\beta + \left[N\int_0^\tau L_{\mathfrak{F}}(s) \parallel \gamma(s) - \vartheta(s) \parallel ds\right]^\beta$$

$$\left[2M_1\sum_{\tau_k<\tau} L_1 \parallel \gamma(\tau_k) - \vartheta(\tau_k) \parallel\right]^\beta.$$

By adopting the equation

$$(x + y \ + z)^\beta \leq 3^{\beta-1}\left[x^\beta + y^\beta + z^\beta\right],$$

Where x, y, z≥ 0, and $\beta > 1$.

$$\gamma(\tau) - \vartheta(\tau) \parallel^\beta \leq 3^{1-\beta}[\epsilon[\varrho(\tau) + \tau\psi]\left[M_0 + Nc_\varrho\right] + N\int_0^\tau L_{\mathfrak{F}}(s) \parallel \gamma(s) - \vartheta(s) \parallel ds$$

$$+2M_1\sum_{\tau_k<\tau} L_1 \parallel \gamma\left(\tau_k - \vartheta(\tau)\right) \parallel]\hat{}\beta$$

$$\parallel \gamma(\tau) - \vartheta(\tau) \parallel^\beta \leq 3^{\frac{1}{\beta}-1}[\epsilon[\varrho(\tau) + \tau\psi]\left[M_0 + Nc_\varrho\right] + N\int_0^\tau L_{\mathfrak{F}}(s) \parallel \gamma(s) - \vartheta(s) \parallel ds$$

$$2M_1\sum_{\tau_k<\tau} L_1 \parallel \gamma\left(\tau_k - \vartheta(\tau_k)\right) \parallel]$$

By using lemma 2.4, is obtained as

$$\parallel \gamma(\tau) - \vartheta(\tau) \leq 3^{\frac{1}{\beta}-1}\left[\epsilon[\varrho(\tau) + \tau\psi]\left[M_0 + Nc_\varrho\right]\left[1 + 3^{\frac{1}{\beta}-1}2M_1L_1\right]^m e^{\left[3^{\frac{1}{\beta}-1} N\int_0^\tau L_{\mathfrak{F}}(s)ds\right]}\right]$$

Then,

Vol. 21, No. 1, (2024)
ISSN: 1005-0930

$$\| \gamma(\tau) - \vartheta(\tau) \|^{\beta} \leq 3^{1-\beta} \left[\epsilon[\varrho(\tau) + \tau\psi]\left[M_0 + Nc_{\varrho}\right]\right]^{\beta} \left[1 + 3^{\frac{1}{\beta}-1} 2M_1 L_1\right]^{m\beta} \left[e^{\left[3^{\frac{1}{\beta}-1} N \int_0^{\tau} L_{\mathfrak{F}}(s)ds\right]}\right]^{\beta}$$

$$\leq 3^{1-\beta}\epsilon^{\beta}[\varrho(\tau) + \tau\psi]^{\beta}\left[M_o + Nc_{\varrho}\right]^{\beta} \left[1 + 3^{\frac{1}{\beta}-1} sM_1 L_1\right]^{m\beta} e^{\left[3^{\frac{1}{\beta}-1}\beta N \int_0^{\tau} L_{\mathfrak{F}}(s)ds\right]}$$

$$\leq V(K_1, L_1, L_{\mathfrak{F}}, \varrho, \psi)\epsilon^{\beta}[\varrho(\tau) + \tau\psi]^{\beta}$$

$$\leq V(K_1, L_1, L_{\mathfrak{F}}, \varrho, \psi)\epsilon^{\beta}\left[\varrho^{\beta}(\tau) + \tau^{\beta}\psi^{\beta}\right],$$

Where

$$V(K_1, L_1, L_{\mathfrak{F}}, \varrho, \psi) = 3^{1-\beta}\left[M_0 + Nc_{\varrho}\right]^{\beta} \left[1 + 3^{\frac{1}{\beta}-1} 2M_1 L_1\right]^{m\beta} e^{\left[3^{\frac{1}{\beta}-1}\beta N \int_0^{\tau} L_{\mathfrak{F}}(s)ds\right]}$$

Therefore, the system is $\beta - HUR$ stable concerning $(\varrho^{\beta}, \psi^{\beta})$.

References

1. D.H.Hyers, G. Isac and T.M.Rassias, Stability of functional equations in several variables, Adv. Appl. Clifford Algebr., (1998).
2. D.H. Hyers. On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA, (1941). Vol.27, pp.222-224.
3. E. Hernandez, Katia A.G. Azevedo and D. O'Regan. On second order differential equations with state-dependent delay, Appl. Anal., (2018). Vol.97 (15), pp.2610-2617.
4. E.Hernandez, K. Balachandran and N. Annapoorani. Existence results for a damped second order abstract functional differential equation with impulses, Math. Comput. Modelling, (2009). Vol.50, pp.1583-1594.
5. T.Li and A.Zada. Connections between Hyers-Ulam stability and uniform exponential stability of discrete evolution families of bounded linear operators over Banach spaces, Adv. Difference Equ, (2016), pp.153.
6. T.Li, A.Zada, and S.Faisal, Hyers-Ulam stability of nth order linear differ ential equations, J. Nonlinear Sci. Appl, (2016). Vol. 9, pp.2070-2075.
7. V.Lakshmikantham, D.D.Bainov and P.S.Simeonov. Theory of Impulsive Differential Equations, World Scientific, Singapore, (1989).
8. Y. Liu and D.O'Regan. Controllability of impulsive functional differential systems with nonlocal conditions, Electron. J. Diff. Equ. (2013). Vol 194, pp.1-10.

9. M.Muslim, A.Kumar and M.Feckan, Existence, uniqueness and stability of solutions to second order nonlinear differential equation with non-instantaneous impulses, Journal of King Saud, (2018). Vol.30, pp.204-213.
10. Y.V.Rogovchenko. Impulsive evolution systems: Main results and new trends, Dyn. Contin. Discrete Impuls Syst, (1997). Vol.3, pp.57-88.
11. T.M.Rassias. On the stability of the linear mapping in Banach spaces, P. Am. Math. Soc, (1978). Vol.72 (2), pp.297-300.
12. A.M.Samoilenko and N.A.Perestyuk. Stability of solutions of differential equations with impulse effect, J. Differ. Equations, (1977). Vol.13, pp.1981-1992.
13. A.M.Samoilenka and N.A.Perestyunk, Impulsive Differential Equations, World Scientific, Singapore, (1995).
14. M.A.Shubov, C.F.Martin, J.P.Dauer and B.Belinskii, Exact controllability of damped wave equation, SIAM J. Control Optim, (1997). Vol.35, pp.1773-1789.
15. S.Tang, A.Zada, S.Faisal, M.M.A.El-Sheikh and T.Li. Stability of higher order nonlinear impulsive differential equations, J. Nonlinear Sci. Appl, (2016). Vol.9, pp.4713-4721.
16. P.Wang, C.Li, J.Zhang, and T. Li. Quasilinearization method for first- order impulsive integro-differential equations, Electron. J. Differential Equa tions, (2019), pp.1-14.
17. J.Wang, M.Feckan and Y.Zhou, Ulam's type stability of impulsive deferential equations, J. Math. Ana. Appl, (2012). Vol.395 (1), pp.258-264.
18. J.Wang, A.Zada and W.Ali. Ulam's-type stability of first-order impulsive differential equations with variable delay in quasi-Banach spaces, Int. J. Nonlin. Sci. Num, (2018). Vol.19 (5), pp.553-560.
19. J.Wang, M.Feckan and Y.Tian. Stability analysis for a general class of non-instantaneous impulsive differential equations, Mediter. J. Math, (2017). Vol.14, pp.1-21.
20. D.Xu and Y.Zhichun. Impulsive delay differential inequality and stability of neural networks. J. Math. Anal. Appl., (2005). Vol.305 (1), pp107-120.
21. X. Yu, J. Wang and Y. Zhang On the β-Ulam-Rassias stability of nonautnonomous impulsive evolution equations, J. Appl. Math. Comput, (2015). Vol.48, pp.461-475.
22. A. Zada, W. Ali and C. Park, Ulam's type stability of higher order nonlinear delay differential equations via integral inequality of Gronwall-Bellman-Bihari's type, Appl. Math. Comput, (2019). Vol.350, pp.60-65.
23. A. Zada, P. Wang, D. Lassoued, and T. Li, Connections between Hyers-Ulam stability and uniform exponential stability of 2-periodic linear nonautonomous systems, Adv. Difference Equ, (2017), pp.192.
24. Hernandez, Eduardo; Andra C. Prokopczyk and Luiz A. C. Ladeira. A Note on State Dependent Partial Functional Differential Equations with Unbounded Delay, Nonlinear Analysis, R.W.A., (2006). Vol. 7(4), pp.510-519.
25. T. Hakon Gronwall. Note on the derivatives with respect to a parameter of the solution of a system of differential equations, Ann. of Math., (1919). Vol. 20 (2), pp.292-296.